# Towards a mathematical definition of Coulomb branches of 3-dimensional N=4 gauge theories

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§0. motivation

$G_c$ : compact Lie group ($G$ : its complexification)
$M$ : a quaternionic representation (symplectic representation of $G$)

$\overset{\text{Physics}}{\rightsquigarrow}$ 3d $N=4$ SUSY gauge theory associated with $(G_c, M)$

$\overset{\text{Physics}}{\rightsquigarrow}$ Moduli Space of vacua

It has two distinguished branches
- $\mathcal{M}_H$ : Higgs branch
- $\mathcal{M}_C$ : Coulomb branch

hyperKähler manifolds with SU(2)-action (rotating cpx structures)

$\mathcal{M}_H$ : mathematically rigorous defined :

$$\mathcal{M}_H = M /\!\!/\!\!/ G_c \quad \text{: hyperKähler quotient}$$

$$= \vec{\mu}^{-1}(0)/G_c \qquad \underset{\text{h.K. moment map}}{\vec{\mu} : M \longrightarrow \mathbb{R}^3 \otimes \mathfrak{g}^*} \qquad = \mu_{\mathbb{C}}^{-1}(0) /\!\!/ G \quad \text{symplectic reduction}$$

$$\vec{\mu} = (\mu_{\mathbb{R}}, \mu_{\mathbb{C}})$$

But there is no mathematically rigorous definition of $\mathcal{M}_C$.

Physicists have many examples of $\mathcal{M}_C$
(or many recipe to determine $\mathcal{M}_C$)

e.g. — moduli space of magnetic monopoles on $\mathbb{R}^3$
— " instantons on $\mathbb{R}^4$ etc
— nilpotent orbits of type A (and conjecturally for classical groups)
$\cap$ Slodowy slice at least

<u>Today</u> Assume $\mathbf{M} = \mathbf{N} \oplus \mathbf{N}^*$ (as $G_{\mathbb{C}}$-module)
(also $G$: connected)

Give a definition of $\mathcal{M}_C$ as an affine variety (scheme)
$\mathcal{M}_C = \operatorname{Spec} \mathcal{A}$
with many interesting properties / structures
e.g. quantization, integrable systems etc

§1. Examples

∘ $\mathbf{N} = 0$ $\quad \mathcal{M}_H = \{0\}$, but $\mathcal{M}_C$ nontrivial (discussed later)

∘ toric hyper Kähler

$$1 \to T^\ell \to \tilde{T} = (\mathbb{C}^\times)^n \to T_F = (\mathbb{C}^\times)^{n-\ell} \to 1, \qquad T^\ell = G, \qquad \tilde{T} \curvearrowright \mathbb{C}^n \text{ natural action}$$

$$\rightsquigarrow \mathcal{M}_H = \mathbb{C}^n \oplus (\mathbb{C}^n)^* /\!/\!/ T_{\mathbb{C}}^\ell$$

$$\mathcal{M}_C = \mathbb{C}^n \oplus (\mathbb{C}^n)^* /\!/\!/ T^\vee_{F,\mathbb{C}} \qquad T_F^\vee : \text{dual torus} \subset \tilde{T}^\vee \cong \tilde{T} \curvearrowright \mathbb{C}^n$$

∘ $\mathbf{N} = \mathfrak{g}$ : adjoint representation

$$\rightsquigarrow \mathcal{M}_H = \mathfrak{g} \oplus \mathfrak{g}^* /\!/\!/ G_c = \{(\alpha, \beta) \in \mathfrak{g} \times \mathfrak{g}^* \mid [\alpha, \beta] = 0\} /\!/ G = \mathfrak{t} \times \mathfrak{t}^* / W \qquad (\mathfrak{g}^* \cong \mathfrak{g})$$

$$\mathcal{M}_C \overset{?}{=} T^* T^\vee / W = \mathfrak{t} \times T^\vee / W \qquad T^\vee = \text{dual torus of } T \subset G$$

cf. Vasserot's construction of DAHA
on equivariant K-theory of the affine Steinberg variety

spherical part of degenerate DAHA, as we use
equivariant homology of the affine Grassmannian Steinberg variety

∘ quiver gauge theory

$Q = (Q_0, Q_1)$ : quiver of type ADE $\qquad$ $G_Q$ = corresponding simple group (adjoint type)

$V$ : $Q_0$-graded vector space

$$G = \prod_i GL(V_i) \curvearrowright N = \bigoplus_{h \in Q_1} \mathrm{Hom}(V_{o(h)}, V_{i(h)})$$

$\rightsquigarrow$ $\mathcal{M}_H = N \oplus N^* /\!/\!/ G_c = \{0\}$ (Lusztig)

$\mathcal{M}_C$ = moduli space of $G_Q$-monopoles on $\mathbb{R}^3$ with charge $= \overrightarrow{\dim} V$

= moduli space of based maps $\mathbb{P}^1 \to$ flag of degree $= \overrightarrow{\dim} V$

$V, W$ : $Q_0$-graded

$G$ : same $\qquad N = \bigoplus_h \mathrm{Hom}(V_{o(h)}, V_{i(h)}) \oplus \bigoplus_i \mathrm{Hom}(V_i, W_i)$

$\rightsquigarrow$ $\mathcal{M}_H = N \oplus N^* /\!/\!/ G_c$ : quiver variety (of type ADE)

$\mathcal{M}_C \overset{?}{=}$ moduli space of $G_Q$-monopoles on $\mathbb{R}^3$ with charge $= \overrightarrow{\dim} V$

singularity at 0 with type $= \overrightarrow{\dim} W$

= slices in affine Grassmannian

(If $\mu = \sum \dim W_i \Lambda_i - \sum \dim V_i \alpha_i$ : dominant)

— subexample : $Q$ : type A $\Rightarrow$ $\mathcal{M}_H \cong \mathcal{S}_\lambda \cap \overline{\mathcal{N}_\mu}$ $\qquad$ $\overline{\mathcal{N}_\mu}$ : nilpotent orbit closure

$\mathcal{M}_C \overset{?}{\cong} \mathcal{S}_{\mu^t} \cap \overline{\mathcal{N}_{\lambda^t}}$ $\qquad$ $\mathcal{S}_\lambda$ : Slodowy slice

∘ Jordan quiver
$$\begin{array}{c} V^{k} \circlearrowleft \\ \uparrow \\ W^{r} \end{array}$$

$\rightsquigarrow$ $\mathcal{M}_H$ = Uhlenbeck space for $U(r)$-instantons on $\mathbb{R}^4$ with charge $k$

$\mathcal{M}_C = S^{k}(\mathbb{R}^4/\mathbb{Z}_r)$ : Uhlenbeck for "$U(1)$-instantons" on $\mathbb{R}^4/\mathbb{Z}_r$ with charge $k$

($\underline{\text{NB}}$ quantization of $\mathcal{M}_C$ : spherical part of cyclotomic rational Cherednik algebra)

More generally

$\mathcal{M}_H$ = quiver variety of affine ADE type $\qquad$ level $= \langle \overrightarrow{\dim} W, \delta \rangle = r$

$\rightsquigarrow$ $\mathcal{M}_C$ = Uhlenbeck space for $G_Q$-instantons on $\mathbb{R}^4/\mathbb{Z}_r$ with charge & repr. at 0 given by $\overrightarrow{\dim} V$

## §2. Definition of $\mathcal{M}_C$

— Reminder of affine Grassmannian and [BFM]

— Step 1°. An infinite dimensional variety $\mathcal{R} \equiv \mathcal{R}_{G,N}$

— 2° Convolution product on $H_*^{G_\mathcal{O}}(\mathcal{R}_{G,N})$ $\quad G_\mathcal{O} = G[[z]]$

$G_\mathcal{K} = G((z))$, $G_\mathcal{O} = G[[z]]$ $\qquad D = \mathrm{Spec}\, \mathbb{C}[[z]] \supset D^\times = \mathrm{Spec}\, \mathbb{C}((z))$

$\mathrm{Gr}_G = G_\mathcal{K}/G_\mathcal{O}$ : affine Grassmann (ind-scheme)

$\cong \{ (\mathcal{P}, \varphi) \mid \mathcal{P} : G\text{-bdle over } D,\ \varphi : \mathcal{P}|_{D^\times} \xrightarrow{\cong} G \times D^\times \text{ trivialization over } D^\times \}$

convolution diagram for $\mathrm{Gr}_G$ :

$$\begin{array}{ccccccc}
\mathrm{Gr}_G \times \mathrm{Gr}_G & \xleftarrow{p} & G_\mathcal{K} \times \mathrm{Gr}_G & \xrightarrow{q} & G_\mathcal{K} \times_{G_\mathcal{O}} \mathrm{Gr}_G & \xrightarrow{m} & \mathrm{Gr}_G \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
([g_1], [g_2]) & \longleftarrow\!\shortmid & (g_1, [g_2]) & \longmapsto & [g_1, [g_2]] & \longmapsto & [g_1 g_2]
\end{array}$$

☆ geometric Satake : $\quad A_1, A_2 \in \mathrm{Perv}_{G_\mathcal{O}}(\mathrm{Gr}_G)$

$$A_1 * A_2 := m_*(q^*)^{-1} p^*(A_1 \boxtimes A_2)$$

$(\mathrm{Perv}_{G_\mathcal{O}}(\mathrm{Gr}_G), *)$ : tensor category $\quad \cong (\mathrm{Rep}\, G^\vee, \otimes)$

☆ [Bezrukavnikov-Finkelberg-Mirković]

$H_*^{G_\mathcal{O}}(\mathrm{Gr}_G) \ni c_1, c_2 \qquad c_1 * c_2 := m_*(q^*)^{-1} p^*(c_1 \boxtimes c_2)$

$H_*^{G_\mathcal{O}}(\mathrm{Gr}_G)$ is a graded commutative algebra with $1 = [e]$

— noncommutative deformation $\quad H_*^{G_\mathcal{O} \rtimes \mathbb{C}^\times}(\mathrm{Gr}_G)$ loop rotation

— integrable system $\quad H^*_{G_\mathcal{O}}(pt) = H^*_G(pt) \longrightarrow H_*^{G_\mathcal{O}}(\mathrm{Gr}_G)$

$\quad\rotatebox{90}{$\cong$}$ polynomial ring (if $G$: connected)

Th. [BFM]

$\mathrm{Spec}\, H_*^{G_\mathcal{O}}(\mathrm{Gr}_G) \longrightarrow \mathrm{Spec}\, H_G^*(pt) = \mathbb{C}^\ell$ is the Kostant–Toda system for $G^\vee$ = Langlands dual group

$T^*G^\vee \curvearrowleft G^\vee \times G^\vee$ left-right multiplication

$\langle e, f, h \rangle$ : $\mathfrak{sl}_2$-triple for regular nilpotent element
$\underset{\mathfrak{n}_+^\vee}{\overset{\cap}{e}}$, $\underset{\mathfrak{n}_-^\vee}{\overset{\cap}{f}}$ $\qquad N_\pm^\vee$ : unipotent group

$\mu_{N_-}^\vee : T^*G^\vee \longrightarrow (\mathfrak{n}_-^\vee \oplus \mathfrak{n}_-^\vee)^*$ : moment map for $N_-^\vee \times N_-^\vee$ -action

Kostant-Toda lattice $= \mu_{N_-}^{\vee\,-1}(e,e) / N_-^\vee \times N_-^\vee \longrightarrow \nu^{-1}(e)/N_-^\vee \cong e + \mathfrak{z}(f) \cong \mathfrak{t}/W$

Kostant slice $\quad \nu : \mathfrak{g}^{\vee *} \to \mathfrak{n}^{\vee *}$

NB. $\quad H_*^{G_\mathcal{O} \rtimes \mathbb{C}^\times}(\mathrm{Gr}_G)$ : quantum Hamiltonian reduction of $\mathrm{Diff}\, G^\vee$

— This is the special case $N = 0$. $\quad (\Rightarrow \mathcal{M}_H = \{0\})$

2nd day

$G$ : connected reductive group $\qquad (N : G\text{-module} \qquad M = N \oplus N^*)$

$G_\mathcal{K} = G((z)) \supset G_\mathcal{O} = G[[z]]$

$\mathrm{Gr}_G = G_\mathcal{K} / G_\mathcal{O}$ : affine Grassmannian

$$\mathrm{Gr}_G \times \mathrm{Gr}_G \xleftarrow{p} G_\mathcal{K} \times \mathrm{Gr}_G \xrightarrow{q} G_\mathcal{K} \times_{G_\mathcal{O}} \mathrm{Gr}_G \xrightarrow{m} \mathrm{Gr}_G$$

$H_*^{G_\mathcal{O}}(\mathrm{Gr}_G)$ is an associative algebra by the convolution product

$$c_1 * c_2 := m_* (q^*)^{-1} p^* (c_1 \boxtimes c_2)$$

- $H_*^{G_\mathcal{O}}(\mathrm{Gr}_G)$ is graded, commutative

$\leftarrow$ loop rotation

- $H_*^{G_\mathcal{O} \rtimes \mathbb{C}^\times}(\mathrm{Gr}_G)$ : noncommutative deformation

- $H_G^*(pt) \longrightarrow H_*^{G_\mathcal{O}}(\mathrm{Gr}_G)$
$\quad \parallel$
$\mathbb{C}[\mathfrak{g}/\!/\mathrm{Ad}\,G] = \mathbb{C}[\mathfrak{t}]^W$

**Th** [BFM] $\operatorname{Spec} H_*^{G_{\mathcal{O}}}(\mathrm{Gr}_G) \longrightarrow \mathfrak{t}/W \cong \mathbb{C}^\ell$

is the Kostant-Toda integrable system for $G^\vee$: Langlands dual group

$\hookrightarrow$ Hamiltonian reduction of $T^*G^\vee$ by $N_-^\vee \times N_-^\vee$ $\qquad N_-^\vee \subset G^\vee$ unipotent

__NB.__ $H_*^{G_{\mathcal{O}} \rtimes \mathbb{C}^\times}(\mathrm{Gr}_G)$ is the quantized Hamiltonian reduction of $\mathrm{Diff}(G^\vee)$

◎ general $N$

$\mathrm{Gr}_G = G_K / G_{\mathcal{O}}$ as before

$$\begin{array}{ccc} \mathcal{T} \equiv \mathcal{T}_{G,N} := G_K \times_{G_{\mathcal{O}}} N_{\mathcal{O}} & \xrightarrow{f} & N_K \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ [g, s] & \longmapsto & gs \end{array}$$

$\swarrow$ $\infty$-rank vector bundle

$\mathrm{Gr}_G$

$$\boxed{\mathcal{R} \equiv \mathcal{R}_{G,N} := f^{-1}(N_{\mathcal{O}})}$$

__Rem.__ $\mathrm{St} = \underset{(\text{nilp. var})}{T^*\mathcal{F}\ell \times T^*\mathcal{F}\ell} \curvearrowleft G$

$= \{(\mathcal{B}_1, x, \mathcal{B}_2) \in \mathcal{F}\ell \times \mathfrak{g} \times \mathcal{F}\ell \mid x \in \operatorname{nilrad}(\operatorname{Lie}\mathcal{B}_1) \cap \operatorname{nilrad}(\operatorname{Lie}\mathcal{B}_2)\}$

Fix $\mathcal{B}_1 = \mathcal{B}$ $\qquad \supset \{(x, \mathcal{B}_2) \in \mathfrak{g} \times \mathcal{F}\ell \mid x \in \operatorname{nilrad}(\operatorname{Lie}\underset{\uparrow \text{fixed}}{\mathcal{B}}) \cap \operatorname{nilrad}(\operatorname{Lie}\mathcal{B}_2)\} = \overline{\mathrm{St}}$

$G \backslash \mathrm{St} = B \backslash \overline{\mathrm{St}}$

Our $\mathcal{R}$ is an analog of $\overline{\mathrm{St}}$

St version : $\mathcal{T} \times_{N_K} \mathcal{T}$

(too $\infty$-dimensional to work)

$$\mathrm{Gr}_G = \bigsqcup_\lambda \mathrm{Gr}_{G,\lambda} \qquad (\lambda : \text{dominant coweight})$$

$\mathrm{Gr}_{G,\lambda}$: $G_\mathcal{O}$-orbit (finite dimensional, smooth)

$$\begin{tikzcd} \mathcal{R} \arrow[r] & \mathrm{Gr}_G \\ \mathcal{T} \arrow[ur] & \end{tikzcd} \qquad \begin{array}{l} \mathcal{R}_\lambda \\ \mathcal{T}_\lambda \end{array} = \text{inverse image of } \mathrm{Gr}_{G,\lambda}$$

☆ $\mathcal{R}_\lambda \to \mathrm{Gr}_{G,\lambda}$ is a vector bundle (of $\infty$-rank)
subbundle of $\mathcal{T}_\lambda$ s.t. $\mathcal{T}_\lambda / \mathcal{R}_\lambda$ : finite rank

Consider equivariant Borel-Moore homology group $H_*^{G_\mathcal{O}}(\mathcal{R})$.

cycles s.t — finite dimensional in base - direction
— finite codimensional in fiber - direction
(relative to $\mathcal{T}$)

The grading is $\mathbb{Z}$-valued (not $\mathbb{Z}_{\geq 0}$)

◎ convolution product $*$ is defined by a similar diagram as in $\mathrm{Gr}_G$.

<u>NB</u> $\quad St = T^*F \times_\mathcal{N} T^*F \Lleftarrow T^*F \times_\mathcal{N} T^*F \times_\mathcal{N} T^*F$

## §3. Properties of $\mathcal{A}$ and $\mathcal{M}_C$

$\mathcal{A} := H_*^{G_\mathcal{O}}(\mathcal{R})$ + convolution product $\qquad \mathcal{M}_C := \mathrm{Spec}\, \mathcal{A}$

1) $\mathcal{A}$ is a $\mathbb{Z}$-graded algebra (finitely generated)
(So $\mathcal{M}_C$ has a $\mathbb{C}^\times$-action.)

unit $1$ = fundamental class of fiber over $[e] \in \mathrm{Gr}_G$

2) $\mathcal{A}$ has a "natural" noncommutative deformation $\mathcal{A}_\hbar$ over $\mathbb{C}[\hbar]$
by $\mathcal{A}_\hbar = H_*^{G_\mathcal{O} \rtimes \mathbb{C}^\times}(\mathcal{R})$

Then $\{\ ,\ \} = \frac{1}{\hbar}[\ ,\ ]\big|_{\hbar=0}$ : Poisson bracket on $\mathcal{A}$ (deg $= -1$
or $-2$)

3) filtration

$$\mathrm{Gr}_G = \bigsqcup_{\lambda:\text{ dominant coweight}} \mathrm{Gr}_{G,\lambda} = \bigcup \mathrm{Gr}_{G,\leq\lambda} \rightsquigarrow \mathcal{R} = \bigcup \mathcal{R}_{\leq\lambda}$$

$\mathrm{Gr}_{G,\leq\lambda}$: closure

<u>Claim.</u> Mayer-Vietories splits $\rightsquigarrow \mathcal{A} = H_*^{G_\mathcal{O}}(\mathcal{R}) = \bigcup H_*^{G_\mathcal{O}}(\mathcal{R}_{\leq\lambda})$

associated graded $\mathrm{gr}\,\mathcal{A} = \bigoplus_\lambda H_*^{G_\mathcal{O}}(\mathcal{R}_\lambda)$

⊙ $\mathrm{gr}\mathcal{A}$ has an explicit presentation. $\mathcal{M}_C \underset{\text{degenerate}}{\rightsquigarrow}$ something combinatorial

$\uparrow$

Q. What is this?

$$\begin{array}{c} \mathcal{R}_\lambda \longrightarrow Gr_{G,\lambda} \longrightarrow G/P_\lambda \\ \cap \\ \mathcal{T}_\lambda \end{array}$$

both vector b'dles

$$\rightsquigarrow \quad H_*^{G_\mathcal{O}}(\mathcal{R}_\lambda) \cong H_{*-2\mathrm{rank}(\mathcal{T}_\lambda/\mathcal{R}_\lambda)}^{G_\mathcal{O}}(Gr_{G,\lambda})$$

$$\cong H^{*+\text{explicit}}_{\mathrm{Stab}(\lambda)}(pt) \qquad \mathbb{C}[\mathfrak{t}]^{W_\lambda}$$

$\Longrightarrow \bigoplus H_*^{G_\mathcal{O}}(\mathcal{R}_\lambda)$ has a base $\{f_p[\mathcal{R}_\lambda]\}$ $\quad f_p$ : base of $H^*_{\mathrm{Stab}(\lambda)}(pt)$

moreover multiplication is

$$f_p[\mathcal{R}_\lambda] * g_\beta[\mathcal{R}_\mu] = f_p g_\beta \, a_{\lambda,\mu}[\mathcal{R}_{\lambda+\mu}]$$

where
$$a_{\lambda,\mu} = \begin{cases} \prod_{\substack{\nu : \text{wt of } N \\ \neq 0}} \nu^{\mathrm{mult}(\nu)\cdot \min(|\langle\lambda,\nu\rangle|, |\langle\mu,\nu\rangle|)} & \text{if } \langle\lambda,\nu\rangle\langle\mu,\nu\rangle < 0 \\ 1 & \text{otherwise} \end{cases}$$

4) $H_G^*(pt) \longrightarrow H_*^{G_\mathcal{O}}(\mathcal{R})$ gives $\mathcal{M}_C \underset{\text{flat}}{\overset{\Phi}{\twoheadrightarrow}} \mathfrak{t}/W \cong \mathbb{C}^\ell$ $\quad \mathfrak{t} = \mathrm{Lie}\,T$

$c \longmapsto c \cap 1$ $\qquad T \subset G$ : max. torus

$W$ : Weyl group

$H_G^*(pt) \cong \mathbb{C}[\mathfrak{g}]^{\mathrm{Ad}G} \cong \mathbb{C}[\mathfrak{t}]^W$

<u>Claim</u>. Over generic point in $\mathfrak{t}/W$ $\left(\mathrm{Spec}(\mathrm{Frac}\, H_G^*(pt))\right)$
(more precisely over the complement of finite union of hyperplanes in $\mathfrak{t}/W$)

$$\mathcal{M}_C\big|_{\mathring{\mathfrak{t}}/W} \cong T^*T^\vee/W\,\big|_{\mathring{\mathfrak{t}}/W} = \mathfrak{t}\times T^\vee/W\,\big|_{\mathring{\mathfrak{t}}/W}$$

$$\Phi \searrow \quad \swarrow p_1$$

$$\mathring{\mathfrak{t}}/W \subset \mathfrak{t}/W$$

(Therefore the integrable system is solved already)

proof) $H_*^{G_\mathcal{O}}(\mathcal{R}) \otimes_{H_G^*(pt)} \mathrm{Frac}\, H_T^*(pt) \underset{\text{localization}}{\cong} H_*(\mathcal{R}^T) \otimes_\mathbb{C} \mathrm{Frac}\, H_T^*(pt)$

Then $\mathcal{R}^T = Gr_T \times N_\mathcal{O}^T$ $\qquad N^T = T$-fixed part of $N$

$Gr_T \cong$ coweight lattice of $T$ $\quad \therefore \mathrm{Spec}\, H_*(\mathcal{R}^T) \cong T^\vee$ : dual torus //

quantization : $\mathbb{C}[\mathfrak{t},\hbar]^W \hookrightarrow \mathcal{A}_\hbar$ : quantized Coulomb branch

Claim. Commutative subalgebra !!
( called ~~Gelfand-Tsetlin~~ Cartan subalgebra )

$\therefore \Phi$ : Poisson commute. Hence $\Phi$ : integrable system

4) $\mathcal{M}_C$ has an action of $\pi_1(G)^\wedge$ : Pontryagin dual of $\pi_1(G)$

$\underset{\mathcal{A}=}{\Leftrightarrow}$ $\mathbb{C}[\mathcal{M}_C]$ has a $\pi_1(G)$-grading

In fact, $\pi_0(\mathcal{R}) = \pi_0(\mathrm{Gr}_G) = \pi_1(G)$ $\therefore \mathcal{R} = \coprod_{\gamma \in \pi_1(G)} \mathcal{R}_\gamma$

$$H_*^{G_\mathcal{O}}(\mathcal{R}_\gamma) * H_*^{G_\mathcal{O}}(\mathcal{R}_{\gamma'}) \longrightarrow H_*^{G_\mathcal{O}}(\mathcal{R}_{\gamma+\gamma'})$$

NB. $G$ : semisimple $\Rightarrow$ $\pi_1(G)$ : finite abelian group
$\pi_1(G)^\vee$ is also

$G = T^\ell \Rightarrow \pi_1(G) \cong \mathbb{Z}^\ell$ $\therefore$ $\pi_1(G)^\vee = T^\ell$

$G = GL \Rightarrow \pi_1(G) \cong \mathbb{Z}$ $\pi_1(G)^\vee = \mathbb{C}^\times$

5) flavor symmetry
Suppose $\exists$
$1 \to G \to \tilde{G} \to G_F \to 1$ s.t. $M = N \oplus N^*$ is a $\tilde{G}$-module

( e.g. $G_F = \prod GL(W_i) \times H_1(\text{graph})$ for quiver gauge theory )

$\Rightarrow \mathcal{M}_C$ has a commutative deformation over $\mathfrak{g}_F /\!/ \mathrm{Ad}\, G_F$

In fact, $\mathcal{R}$ has a $\tilde{G}_\mathcal{O}$-action

$\rightsquigarrow$ $H_*^{\tilde{G}_\mathcal{O}}(\mathcal{R}) \leftarrow H^*_{\tilde{G}}(pt) \leftarrow H^*_{G_F}(pt)$ ) Spec

$\tilde{\mathcal{M}}_C \longrightarrow \tilde{\mathfrak{g}} /\!/ \mathrm{Ad}\, \tilde{G} \to \mathfrak{g}_F /\!/ \mathrm{Ad}\, G_F$ fiber over $0$ = original $\mathcal{M}_C$

One can also construct a (partial) resolution of singularities
for each dominant coweight $\lambda_F : \mathbb{C}^* \to G_F$

In fact, consider $\tilde{\mathcal{R}} = \mathcal{R}_{\tilde{G},N} \longrightarrow \mathrm{Gr}_{\tilde{G}} \longrightarrow \mathrm{Gr}_{G_F}$ fiber over $[e] \in \mathrm{Gr}_{G_F}$ = original $\mathcal{R}$

Use the stratification on $\mathrm{Gr}_{G_F}$, to introduce a filtration
on $H_*^{\tilde{G}_\mathcal{O}}(\tilde{\mathcal{R}})$. Then take the associated graded.

Braden-Licata-Proudfoot-Webster : Symplectic duality

quantization of $\mathcal{M}_C$ $\xleftrightarrow[\text{Koszul dual}]{}$ quantization of $\mathcal{M}_H$

under some conditions ($\mathcal{M}_C$ was not defined in [BLPW])

§4 (Conjectural) "duality" between $\mathcal{M}_H$ and $\mathcal{M}_C$. (More elementary than [BLPW])

1) stratum

<u>Fact.</u> $\mathcal{M}_H$ has a stratification (symplectic leaves)

$$\mathcal{M}_H = \bigsqcup_{\alpha \in A} \mathcal{M}_H^\alpha$$

$A = \{$conjugacy classes of stabilizers$\}$

<u>Conjecture</u> $\mathcal{M}_C$ has a stratification parametrized by the same set $A$ : $\mathcal{M}_C = \bigsqcup_{\alpha \in A} \mathcal{M}_C^\alpha$

with the opposite closure relation

(e.g., $\mathcal{M}_H = \{0\} \Rightarrow \mathcal{M}_C$ : smooth symplectic manifold)

moduli space of vacua $= \bigsqcup_{\alpha \in A} \mathcal{M}_C^\alpha \times \mathcal{M}_H^\alpha$

2) $\mathbb{C}^\times$-actions

$\underset{t}{\mathbb{C}^\times \curvearrowright M}$ or $\underset{t \quad 1}{N \oplus N^*}$ $\rightsquigarrow \mathbb{C}^\times \curvearrowright \mathcal{M}_H$

Then $\mathbb{C}[\mathcal{M}_H] = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathbb{C}[\mathcal{M}_H]_n$ & grade $0 = \mathbb{C}$ (often) (i.e. $\mathcal{M}_H$ is cone)

On the other hand, $\mathcal{M}_C$ is not cone in general

<u>Conjecture</u> $\mathcal{M}_C$ is cone $\overset{?}{\Longleftrightarrow}$ $\mu_{\mathbb{C}}^{-1}(0) \subset M$ is complete intersection

3) Group action and deformation/resolution
(mass parameter) (Kähler parameter)

- flavor symmetry $1 \to G \to \tilde{G} \to G_F \to 1$

$\mathrm{Hom}(\mathbb{C}^*, G_F) \ni \lambda_F \rightsquigarrow \mathbb{C}^\times \xrightarrow{\lambda_F} G_F \curvearrowright \mathcal{M}_H = M /\!/\!/ G$

$\rightsquigarrow$ deformation / resolution of $\mathcal{M}_C$

- $\mathrm{Hom}_{grp}(G, \mathbb{C}^*) \ni \chi \rightsquigarrow \mu_{\mathbb{C}}^{-1}(0) /\!/_{\chi} G = \mathrm{Proj}\left(\bigoplus_{n \geq 0} \mathbb{C}[\mu_{\mathbb{C}}^{-1}(0)]^{G, \chi^n}\right) \longrightarrow \mathcal{M}_H$

<u>Note</u> $\|\wr$ often (partial) resolution

$\mathrm{Hom}_{grp}(\mathbb{C}^*, \pi_1(G)^\wedge) \ni \chi \rightsquigarrow \mathbb{C}^\times \xrightarrow{\chi} \pi_1(G)^\wedge \curvearrowright \mathcal{M}_C$ group action

Thus mass / Kähler parameter are exchanged between $\mathcal{M}_C$ and $\mathcal{M}_H$.

<u>Conjecture</u> $\underset{\chi}{\lambda_F}$ has fixed points only $\{0\}$ on $\underset{\mathcal{M}_C}{\mathcal{M}_H}$

$\Longleftrightarrow$ $\underset{\chi}{\lambda_F}$ gives a resolution (orbifold in general) of $\underset{\mathcal{M}_H}{\mathcal{M}_C}$